Chapter 3.

Antidifferentiation.

As mentioned in the Preliminary Work section at the beginning of this text:
The illustrations of calculator displays shown in this text may not exactly match the displays from your calculator. The illustrations are not included to show what your calculator will necessarily display but are more to inform you that, at that moment, the use of your calculator could well be appropriate.

Antidifferentiation.

As we were reminded in the *Preliminary work* section at the beginning of this text, **antidifferentiation** (also called **integration** for a reason we will see later in this book) is the opposite of differentiation.

The derivative of x^2 is $2x$.
The antiderivative of $2x$ is $x^2 + c$.

Using the integration symbol we could write this as:

$$\int 2x \, dx = x^2 + c$$

The "+ c" is needed because all functions of the form $x^2 + c$ differentiate to $2x$, for example:

The derivative of $x^2 + 1$ is $2x$. The derivative of $x^2 + 3$ is $2x$.
The derivative of $x^2 - 3$ is $2x$. The derivative of $x^2 - 7$ is $2x$.

We say that the *family* of curves $y = x^2 + c$ all have the same gradient function.
Given further information it may be possible to determine the value of the constant, c, as we will see in example 2.

Antidifferentiating powers of x.

The *Preliminary work* section also reminded us of the rule:

$$\text{If } \frac{dy}{dx} = ax^n \quad \text{then } y = \frac{ax^{n+1}}{n+1} + c$$

This can be remembered as:

"*Increase the power by one and divide by the new power.*"

Using the integration symbol this rule is written:

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c$$

Because integrals of this form involve an unknown constant they are called **indefinite integrals.**

Example 1

Find the antiderivative of each of the following.

(a) x^4 (b) $6x^3$ (c) 7 (d) $8x + \sqrt{x}$

(a) If $\frac{dy}{dx} = x^4$

then $y = \frac{x^5}{5} + c$

The antiderivative is $\frac{x^5}{5} + c$.

(b) If $\frac{dy}{dx} = 6x^3$

then $y = \frac{6x^4}{4} + c$

The antiderivative is $\frac{3x^4}{2} + c$.

(c) If $\frac{dy}{dx} = 7$ (i.e. $7x^0$)

then $y = \frac{7x^1}{1} + c$

The antiderivative is $7x + c$.

(d) If $\frac{dy}{dx} = 8x + x^{\frac{1}{2}}$

then $y = \frac{8x^2}{2} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

The antiderivative is $4x^2 + \frac{2}{3}x^{\frac{3}{2}} + c$.

The reader should ① confirm that differentiating each of the above answers does give the required gradient function

and ② confirm these answers using a calculator. Remember, calculators tend not to include the "+ c" (we need to remember to include the constant ourselves when writing an answer obtained from a calculator) and the displays may feature spaces for entries to be made above and below the integral sign, as shown below right. This is for *definite integrals*, a concept we will meet later in this book. For the moment simply leave such entries empty.

$$\int (x^4)\,dx \qquad \frac{x^5}{5}$$
$$\int (6x^3)\,dx \qquad \frac{3\cdot x^4}{2}$$
$$\int (7)\,dx \qquad 7\cdot x$$
$$\int (8x+\sqrt{x})\,dx \qquad 4\cdot x^2 + \frac{2\cdot x^{\frac{3}{2}}}{3}$$

$$\int_{\square}^{\square} x^4\,dx \qquad \frac{x^5}{5}$$
$$\int_{\square}^{\square} 6x^3\,dx \qquad \frac{3\cdot x^4}{2}$$
$$\int_{\square}^{\square} 7\,dx \qquad 7\cdot x$$

Notice that the results for example 1 support the following statements:

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

and for constant, a,

$$\int a\,f(x)\,dx = a\int f(x)\,dx$$

These statements together define the **linearity property** of antidifferentiation.

Example 2

If $\frac{dy}{dx} = 4x - \frac{2}{x^2}$ and, when $x = -1, y = 2$, find (a) y in terms of x,

(b) y, when $x = 0{\cdot}5$.

(a) If

$$\frac{dy}{dx} = 4x - 2x^{-2}$$

$$y = 4\frac{x^2}{2} - \frac{2x^{-1}}{-1} + c$$

$$= 2x^2 + \frac{2}{x} + c$$

We are told that when $x = -1, y = 2$.

Thus $2 = 2(-1)^2 + \frac{2}{-1} + c$

i.e. $2 = c$

$\therefore$ $y = 2x^2 + \frac{2}{x} + 2$

(b) If $x = 0{\cdot}5$, $y = 2(0{\cdot}5)^2 + \frac{2}{0{\cdot}5} + 2$

$= 6{\cdot}5$

When $x = 0{\cdot}5, y = 6{\cdot}5$.

Exercise 3A

Find the antiderivative of each of the following.

1.	x^6	2.	x^3	3.	$10x^4$
4.	$7x^2$	5.	$8x$	6.	8
7.	$\sqrt{x}$	8.	$\sqrt[3]{x}$	9.	$x^{\frac{5}{2}}$
10.	$6x^{\frac{3}{2}}$	11.	$4x^{-\frac{1}{2}}$	12.	$\frac{4}{\sqrt{x}}$
13.	$\frac{10}{x^4}$	14.	$-\frac{9}{x^2}$	15.	$-\frac{16}{\sqrt{x}}$
16.	$6x^2 - 4x + 3$	17.	$12x^3 + 3$	18.	$x^3 + 3x^2 + 2x$
19.	$1 + 4x + 18x^2$	20.	$3\sqrt{x} + 6x$	21.	$3x^2 + 14x + 8$

22. $(3x + 2)(x + 4)$

23. $(x - 2)(x + 6)$

24. $(3x - 2)(3x + 2)$

25. $4x(3x^2 + 3)$

26. $\dfrac{4x^2 + 5x}{x}$

27. $\dfrac{2x + 1}{x^3}$

28. $\dfrac{6x + 4}{\sqrt{x}}$

29. $\dfrac{1 - x^2}{\sqrt{x}}$

30. $\dfrac{\sqrt{x}}{x} + 1$

31. Find y in terms of x given that $\dfrac{dy}{dx} = 6x^2 + 1$ and $y = 13$ when $x = 2$.

32. Find y in terms of x given that $\dfrac{dy}{dx} = 4x - 3$ and $y = 29$ when $x = -3$.

33. Find A in terms of t given that $\dfrac{dA}{dt} = 1 - \dfrac{6}{t^2}$ and $A = -2$ when $t = 2$.

34. Find v in terms of x given that $\dfrac{dv}{dx} = x + \dfrac{1}{\sqrt{x}}$ and $v = 2$ when $x = 4$.

35. If $f'(x) = \dfrac{6x^2}{5} - \dfrac{5}{6x^2}$ and $f(5) = 51$ find (a) $f(x)$, (b) $f(1)$, (c) $f(-1)$.

Further antidifferentiation.

Suppose we are asked to find the antiderivative of $(x + 3)^5$.
We could expand the bracket and then antidifferentiate each term but this would be a tedious process to do "by hand".

Using a calculator, and remembering to include the "+c", we find that:

The antiderivative of $(x + 3)^5$ is $\dfrac{(x+3)^6}{6} + c$.

Can we simply apply the rule "*Increase the power by one and divide by the new power*"?

Were we to apply this rule in an attempt to antidifferentiate $(2x + 3)^5$ it would suggest an answer of $\dfrac{(2x+3)^6}{6} + c$.

$$\int (x+3)^5\, dx \qquad \frac{(x+3)^6}{6}$$

$$\int (2\cdot x+3)^5\, dx \qquad \frac{(2\cdot x+3)^6}{12}$$

However, as the display shows, this is not the correct answer. Instead the antiderivative of $(2x + 3)^5$ is $\dfrac{(2x+3)^6}{12} + c$.

Why did "*Increase the power by one and divide by the new power*" work for the first case but not the second?

To answer this question consider what happens when we differentiate $(2x + 3)^6$.

If $\quad y = (2x+3)^6 \quad$ then $\quad \frac{dy}{dx} = 6(2x+3)^5 \times 2$

$$= 12(2x+3)^5$$

Notice that the derivative of $(2x + 3)$ also appears in the answer because, as we saw when using the chain rule:

$$\text{If } y = [f(x)]^n \text{ then } \frac{dy}{dx} = n\,[f(x)]^{n-1}\,\mathbf{f'(x)}$$

From this it follows that:

$$\text{If } \frac{dy}{dx} = f'(x)\,[f(x)]^n \quad \text{then } y = \frac{[f(x)]^{n+1}}{n+1} + c$$

This is very similar to the rule for antidifferentiating ax^n, i.e. "*Increase the power by one and divide by the new power*" the important difference being that we need the derivative of the function that is raised to a power to be present too.

Using the integration symbol this rule is written:

$$\int f'(x)\,[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

Note
- In most of the following examples two methods of solution are shown.

  In "method one" the approach is to make an intelligent first attempt at the antiderivative, differentiate it, and then use the result to adjust the first attempt appropriately. If we are attempting to antidifferentiate an expression that is of the form

  $$f'(x)\,[f(x)]^n$$

  or some scalar multiple thereof, our initial attempt should be of the form $f(x)^{n+1}$.

  In "method two" the given expression is first manipulated so that the task becomes that of determining

  $$a\int f'(x)\,[f(x)]^n\,dx$$

  from which the answer, $a\frac{[f(x)]^{n+1}}{n+1} + c$, follows.

  The reader should be able to follow both methods but is advised to "adopt" whichever one they prefer.

- Do not expect all of the questions to involve expressions of the form

  $$f'(x)\,[f(x)]^n.$$

  Example 4 part (a), $x(6x-1)^2$, is not of this form for example.

Example 3

Antidifferentiate (a) $(2x+1)^4$, (b) $\dfrac{12}{(1-3x)^2}$.

(a) Method one. (Making an intelligent guess then adjusting.)

Noticing that $(2x+1)^4$ is of the form $f'(x)\,[f(x)]^n$, except for a suitable scalar multiple, we try $(2x+1)$ to "the next power up".

$$\begin{aligned} \text{If} \quad y &= (2x+1)^5 \\ \text{then} \quad \frac{dy}{dx} &= 5(2x+1)^4(2) \\ &= 10(2x+1)^4 \end{aligned}$$

Our initial trial needs to be divided by ten.

The required antiderivative is:

$$\frac{1}{10}(2x+1)^5 + c.$$

Method two. (Rearranging.)

(First note that the derivative of $2x+1$ is 2.)

Rearranging $(2x+1)^4$ into the form $a\,f'(x)\,[f(x)]^n$:

$$\begin{aligned} \int (2x+1)^4\,dx &= \int \frac{1}{2}\times 2\times(2x+1)^4\,dx \\ &= \frac{1}{2}\times\int 2\times(2x+1)^4\,dx \\ &= \frac{1}{2}\times\frac{(2x+1)^5}{5} + c \\ &= \frac{(2x+1)^5}{10} + c \end{aligned}$$

The reader should also ensure they can obtain this same antiderivative using a calculator.

$\int (2\cdot x+1)^4\,dx$

$$\frac{(2\cdot x+1)^5}{10}$$

(b) Method one. (Making an intelligent guess then adjusting.)

Noticing that $12(1-3x)^{-2}$ is of the form $f'(x)\,[f(x)]^n$, except for a suitable scalar multiple, we try $(1-3x)$ to "the next power up".

I.e. given $\frac{dy}{dx} = 12(1-3x)^{-2}$ we try $y = (1-3x)^{-1}$

If $y = (1-3x)^{-1}$

then
$$\frac{dy}{dx} = (-1)(1-3x)^{-2}(-3)$$
$$= 3(1-3x)^{-2}$$

Our initial trial needs to be multiplied 4.

The required antiderivative is

$$\frac{4}{1-3x} + c.$$

Method two. (Rearranging.)

(First note that the derivative of $1-3x$ is -3.)

Rearranging $12(1-3x)^{-2}$ into the form $a\,f'(x)\,[f(x)]^n$:

$$\int 12(1-3x)^{-2}\,dx = \int (-4)\times(-3)\times(1-3x)^{-2}\,dx$$
$$= -4\times\int(-3)\times(1-3x)^{-2}\,dx$$
$$= -4\times\frac{(1-3x)^{-1}}{-1} + c$$
$$= \frac{4}{1-3x} + c$$

Again the reader should ensure they can obtain this same antiderivative using a calculator.

$$\int \frac{12}{(1-3\cdot x)^2}\,dx$$
$$\frac{-4}{(3\cdot x-1)}$$

Example 4

Antidifferentiate (a) $x(6x-1)^2$, (b) $x(6x^2-1)^2$.

(a) The derivative of $6x-1$ is 6. Hence $x(6x-1)^2$ is **not** a scalar multiple of something of the form $f'(x)\,[f(x)]^n$.

In this case we expand the expression and then antidifferentiate.

$$\begin{aligned} \text{If} \quad \frac{dy}{dx} &= x(6x-1)^2 \\ &= x(36x^2-12x+1) \\ &= 36x^3-12x^2+x \\ \text{then} \quad y &= 9x^4-4x^3+\frac{x^2}{2}+c \end{aligned}$$

The required antiderivative is $9x^4-4x^3+\frac{x^2}{2}+c$.

(b) We could again choose to expand $x(6x^2-1)^2$ and antidifferentiate each term or, noticing that we have an expression of the form $f'(x)\,[f(x)]^n$, except for a suitable scalar multiple, we could proceed as in example 3.

<u>Method one.</u> (Making an intelligent guess then adjusting.)

Given $\frac{dy}{dx} = x(6x^2-1)^2$ we try $y = (6x^2-1)^3$

$$\begin{aligned} \text{If} \quad y &= (6x^2-1)^3 \\ \text{then} \quad \frac{dy}{dx} &= (3)(6x^2-1)^2(12x) \\ &= 36x(6x^2-1)^2 \end{aligned}$$

We need to ÷ our initial trial by 36.

The required antiderivative is

$$\frac{(6x^2-1)^3}{36}+c.$$

$$\int x\cdot(6\cdot x-1)^2dx \qquad 9\cdot x^4-4\cdot x^3+\frac{x^2}{2}$$

$$\int x\cdot(6\cdot x^2-1)^2dx \qquad \frac{(6\cdot x^2-1)^3}{36}$$

<u>Method two.</u> (Rearranging.)

$$\begin{aligned} \int x(6x^2-1)^2\,dx &= \int \frac{1}{12}\times(12x)\times(6x^2-1)^2\,dx \\ &= \frac{1}{12}\times\int (12x)\times(6x^2-1)^2\,dx \\ &= \frac{1}{12}\times\frac{(6x^2-1)^3}{3}+c \\ &= \frac{(6x^2-1)^3}{36}+c \end{aligned}$$

Example 5

If $\dfrac{dA}{dt} = 2(5 + 4t)^3$ find A in terms of t given that when $t = -1, A = 1$.

Method one. (Making an intelligent guess then adjusting.)

Noticing that $2(5 + 4t)^3$ is of the form $f'(x)\,[f(x)]^n$, except for a suitable scalar multiple, we try $(5 + 4t)$ to "the next power up".

$$\begin{aligned} \text{If} \qquad A &= (5 + 4t)^4 \\ \text{then} \qquad \frac{dA}{dt} &= 4(5 + 4t)^3(4) \\ &= 16(5 + 4t)^3 \end{aligned}$$

Our initial trial must be divided by 8. $\qquad \therefore \quad A = \dfrac{(5 + 4t)^4}{8} + c.$

Now when $t = -1, A = 1$.

$$\therefore \qquad 1 = \frac{(5 - 4)^4}{8} + c \qquad \text{giving} \qquad c = \frac{7}{8}.$$

$$\text{Thus} \qquad A = \frac{(5 + 4t)^4}{8} + \frac{7}{8}.$$

Method two. (Rearranging.)

$$\begin{aligned} \int 2(5 + 4t)^3\, dt &= \int \frac{1}{2} \times 4 \times (5 + 4t)^3\, dt \\ &= \frac{1}{2} \times \int 4 \times (5 + 4t)^3\, dt \\ &= \frac{1}{2} \times \frac{(5 + 4t)^4}{4} + c \\ \therefore \qquad A &= \frac{(5 + 4t)^4}{8} + c \end{aligned}$$

$\int 2\cdot(5 + 4\cdot t)^3 dt \qquad \dfrac{(4\cdot t + 5)^4}{8}$

The constant is found as in method 1, i.e.:

When $t = -1, A = 1$.

$$\therefore \qquad 1 = \frac{(5 - 4)^4}{8} + c \qquad \text{giving} \qquad c = \frac{7}{8}.$$

$$\text{Thus} \qquad A = \frac{(5 + 4t)^4}{8} + \frac{7}{8}.$$

Example 6

Find y in terms of x given that $\frac{dy}{dx} = 72x(3x^2 - 1)^5$ and $y = 100$ when $x = 1$.

Method one. (Making an intelligent guess then adjusting.)

Noticing that $72x(3x^2 - 1)^5$ is of the form $f'(x)\,[f(x)]^n$, except for a suitable scalar multiple, we try $(3x^2 - 1)^6$.

If $\quad y = (3x^2 - 1)^6$

then

$$\frac{dy}{dx} = 6(3x^2 - 1)^5(6x) = 36x(3x^2 - 1)^5$$

Our initial trial must be multiplied by 2.

$$\therefore \quad y = 2(3x^2 - 1)^6 + c.$$

Now when $x = 1$, $y = 100$.

$$\therefore \quad 100 = 2(2)^6 + c \quad \text{giving} \quad c = -28.$$

Thus $\quad y = 2(3x^2 - 1)^6 - 28.$

Method two. (Rearranging.)

$$\int 72x(3x^2 - 1)^5\,dx = \int 12 \times 6x \times (3x^2 - 1)^5\,dx$$

$$= 12 \times \int 6x \times (3x^2 - 1)^5\,dx$$

$$= 12 \times \frac{(3x^2 - 1)^6}{6} + c$$

$$\therefore \quad y = 2(3x^2 - 1)^6 + c$$

The constant is found as in method 1, i.e.:
When $x = 1$, $y = 100$.

$$\therefore \quad 100 = 2(2)^6 + c \quad \text{giving} \quad c = -28.$$

Thus $\quad y = 2(3x^2 - 1)^6 - 28.$

$$\int 72 \cdot x \cdot (3 \cdot x^2 - 1)^5 dx$$

$$2 \cdot (3 \cdot x^2 - 1)^6$$

Note: With practice you may find you are able to write the antiderivative directly, without formally presenting either method.

Exercise 3B

Find the antiderivative of each of the following.
(Try to do most without the assistance of a calculator.)

1. $(3x+2)^3$	2. $(3x+2)^4$	3. $x(3x+2)$
4. $(1+5x)^4$	5. $(1-5x)^3$	6. $10x(x^2+5)^4$
7. $20x(x^2-7)^4$	8. $x(1+5x)^2$	9. $(2x+1)^2$
10. $x(2x+1)^2$	11. $(5x+1)^3$	12. $21(5-7x)^3$
13. $16(2x+1)^3$	14. $45(3x-2)^4$	15. $(2x-1)(x^2-x+3)^4$
16. $48(6x+1)^3$	17. $2(5x+1)^3$	18. $150(x-1)(3x^2-6x+1)^4$
19. $5(3x-1)^4$	20. $3(9x+1)^2$	21. $x(3x+4)$
22. $2(3x-1)^2$	23. $2x(x-1)^2$	24. $(x+1)(x-1)$
25. $(1+x)^3$	26. $(1-x)^3$	27. $x(1+x)$
28. $2x(1+x)^2$	29. $12x(1+x^2)^2$	30. $2x(1+x^2)^6$
31. $-24(1-2x)^3$	32. $54(2x-1)^8$	33. $15(5-6x)^4$
34. $(3-2x)^3$	35. $6(2x-3)^8$	36. $12(5-6x)^3$
37. $(2x+1)(x^2+x+3)^4$	38. $20x(5x^2+3)^7$	39. $(1-2x)(x^2-x+3)^4$
40. $\dfrac{1}{(x+2)^4}$	41. $\dfrac{5}{(x+1)^2}$	42. $(1-x)(x^2-2x+1)^3$
43. $\dfrac{2}{(x+3)^3}$	44. $\dfrac{18x}{(x^2-3)^4}$	45. $\dfrac{1}{(x-2)^2}$
46. $\dfrac{1}{(2x-1)^2}$	47. $\dfrac{20}{(3-2x)^3}$	48. $10(6x-1)(3x^2-x+1)^4$
49. $-\dfrac{1}{(x-2)^3}$	50. $\dfrac{12}{(3x-1)^2}$	51. $\dfrac{20}{(1-5x)^3}$
52. $\sqrt{3x+2}$	53. $12\sqrt{2x-5}$	54. $\dfrac{6}{\sqrt{1+2x}}$
55. $1+(1-5x)^2$	56. $12\sqrt[3]{3x-2}$	57. $1+x(1-5x)^2$

58. $\dfrac{12}{(2x-3)^4}$

59. $12(2x+1)^2 + 9(3x-2)^2$

60. $\sqrt{x+3} + \sqrt{x+1}$

61. $\dfrac{10x+15}{\sqrt{x^2+3x-1}}$

62. Find A in terms of p given that $\dfrac{dA}{dp} = 6(p+1)^2$ and $A = 21$ when $p = 1$.

63. Find y in terms of x given that $\dfrac{dy}{dx} = 20(2x+1)^4$ and $y = 25$ when $x = 0$.

64. Find f(x) given that $f'(x) = 32(3-2x)^3$ and f(2) = 1.

65. Find y in terms of x given that $\dfrac{dy}{dx} = 15x(5x^2-1)^2$ and $y = 40$ when $x = 1$.

66. Find v in terms of t given that $\dfrac{dv}{dt} = \dfrac{100t}{(t^2+1)^3}$ and when $t = 2$, $v = 7$.

67. Find x in terms of t given that $\dfrac{dx}{dt} = -\dfrac{10}{(2t+1)^2}$ and $x = 2$ when $t = -1$.

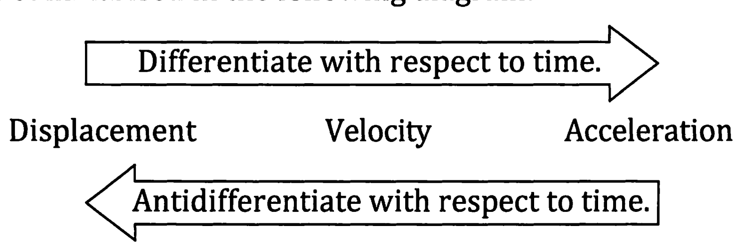
68. If $\dfrac{dy}{dx} = 24(2x-1)^3$ and $y = 5$ when $x = 0$ find
    (a) y in terms of x,
    (b) y, when $x = 1$,
    (c) x, when $y = 245$.

Rectilinear motion.

From our understanding of antidifferentiation as the opposite of differentiation it follows that antidifferentiating, or integrating, velocity with respect to time gives displacement and that antidifferentiating, or integrating, acceleration with respect to time gives velocity.

i.e. $x = \int v\,dt$ and $v = \int a\,dt$

These facts are summarised in the following diagram:



Remember that each antidifferentiation will introduce a constant which, given sufficient information, may be determined.

Example 7

A particle travels along a straight line with its velocity at time t seconds given by v m/sec where $v = 3t^2 + 2$.
The initial displacement of the particle from a point O on the line is ten metres.
Find (a) the acceleration when $t = 4$,
(b) the displacement from O when $t = 5$.

(a) $v = 3t^2 + 2$

Now $a = \frac{dv}{dt}$

$= 6t$

When $t = 4$ the acceleration is 24 m/s^2.

(b) $x = \int v \, dt$

$= \int (3t^2 + 2)\, dt$

$= t^3 + 2t + c$

We know that initially, i.e. when $t = 0$, $x = 10$.

$\therefore \quad 10 = (0)^3 + 2(0) + c \qquad$ i.e. $c = 10$.

Thus $x = t^3 + 2t + 10 \qquad \therefore$ When $t = 5$, $x = 145$.

When $t = 5$ the displacement from O is 145 metres.

Exercise 3C

Questions 1 to 9 all involve rectilinear motion with x metres, v m/s and a m/s^2 the displacement, velocity and acceleration of a body, with respect to an origin O, at time t seconds.

1. If $v = 6t^2 + 4$ find (a) the acceleration when $t = 4$,
   (b) the displacement when $t = 2$ given that for $t = 1$, $x = 5$.

2. If $a = 6t - 2$ and when $t = 0$ the velocity is 1 m/s and the displacement is 5m, find
   (a) the acceleration when $t = 1$,
   (b) the velocity when $t = 2$,
   (c) the displacement when $t = 3$.

3. If $a = 2t(5 - 6t)$ and the body is initially at O and moving with velocity 2 m/s find
   (a) the velocity when $t = 2$,
   (b) the speed when $t = 2$,
   (c) the displacement when $t = 3$.

4. A body has an initial displacement of 5 m and velocity 2 m/s. Find the displacement and velocity after 4 seconds given that $a = \dfrac{6}{(t+1)^3}$.

5. If $v = \dfrac{1}{(t+1)^2}$ and when $t = 0$, $x = 3$, find x when $t = 4$.

6. A body is initially at rest at O and moves such that $a = 2 + \sqrt{t}$.
   Find (a) the velocity when $t = 9$,
   (b) the displacement when $t = 9$.

7. If $x = 5t + \dfrac{4}{t}$, $t > 0$, find the displacement when the velocity is 4 m/s.

8. If $a = 8 - t$, $t \geq 0$, find the displacement when the velocity is 2 m/s given that when $t = 0$, $x = 16$ and $v = 20$.

9. If $a = \dfrac{48(2t+1)^2}{5}$ and when $t = 1$, $v = 44$ and $x = 19$, find expressions for v and x as functions of t.

10. A body is initially at an origin, O, and at that instant the body has a velocity of 14 m/s. The acceleration, t seconds later, is $(3t - 11)$ m/s^2. Find the velocity of the body when it is **next** at O.

11. A body is initially at rest at an origin, O, and moves in a straight line such that its acceleration, t seconds later, is $(18 - 6t)$ m/s^2.
    Find (a) the value of t when the body is next at rest and the displacement at this time,
    (b) the distance the body moves from $t = 5$ to $t = 7$.

12. A train leaves a station and accelerates from rest at a constant 0·25 m/s^2 for two minutes. (a) How far does the train travel in this time?
    (b) What is the velocity of the train at the end of the two minutes?

13. A body is moving in a straight line with constant acceleration a m/s^2. Use calculus to obtain expressions for v, the velocity of the body in m/s, and s, the displacement of the body in metres, at time t, given that when $t = 0$, $s = 0$ and $v = u$.

14. A particle travels along a straight line such that its acceleration at time t seconds is equal to $(6t + 1)$ m/s^2. When $t = 2$ the displacement is 12 metres and when $t = 3$ the displacement is 34 metres. Find the displacement and velocity when $t = 4$.

15. A particle travels along a straight line with its acceleration at time t secs equal to $(3t + 2)$ m/s^2. The particle has an initial positive velocity and travels 30 m in the fourth second. Find the velocity of the body when $t = 5$.

Miscellaneous Exercise Three.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

Use the product rule to determine $\frac{dy}{dx}$ for each of the following.

1. $y = (x + 3)(x^2 + 1)$
2. $y = (x - 5)(x^2 - 7)$
3. $y = (x + 1)(x^2 + x + 1)$
4. $y = (2x - 3)(x^2 + 5)$
5. $y = (3x - 2)(3x^2 + x - 1)$
6. $y = (4x + 1)(x^2 - 5x + 1)$

7. If $f(x) = (2x - 3)^5$ find, without the assistance of your calculator:

   (a) $f'(x)$ (b) $f'(2)$ (c) $f''(x)$ (d) $f''(2)$

8. Given that $y = ax^3 + x^2 + bx + 3$, $y'(2) = 50$ and $y''(1) = 23$ find the values of the constants a and b.

9. The displacement of a body from an origin O, at time t seconds, is x metres where

$$x = 2t^3 - 9t^2 + 5, \quad t \geq 0.$$

   Find the displacement and the velocity of the body from O when the acceleration is zero.

10. Find the coordinates of the points on the graphs of the given functions where the gradient is as stated.

    (a) $y = 3x^2 + 2x$, gradient = −10.

    (b) $y = x^3 - 5x$, gradient = 43.

    (c) $y = \frac{5 - x}{3x + 1}$, gradient = − 1.

11. A rocket is launched from its pad at ground level and moves vertically upwards with its engines causing it to accelerate at $(28 - 0{\cdot}2t)$ m/s^2 for the first two minutes, t being the time, in seconds, since the launch.

    At the end of the two minutes the engines reduce the thrust they produce to a level just sufficient to maintain the vertical velocity achieved at that time.

    How high is the rocket after (a) 1 minute,

    (b) 2 minutes,

    (c) 3 minutes?

12. (Without the use of your calculator.)
The cost function for a particular item is given by \$C where C is given by

$$C = \frac{1}{12}n^3 - 12n^2 + 800n + 1000$$

and n is the number of such items produced.
Find the marginal cost for $n = 100$ and explain what this tells you about the cost of producing one more item at this level of production.

13. The owner of a garden centre wishes to fence a rectangular area of 300 m^2. She wishes to fence three sides with fencing that costs \$9 per metre and the fourth side with fencing costing \$15 per metre. Find the dimensions of the rectangular area that will minimise her fencing costs.

14. A purification unit takes used coolant and removes the impurities that have become dissolved in it. The cost, \$$C$, of removing p% of the impurities from one tonne of coolant is given by $C = \frac{1000p}{100 - p}$.

(a) Find to the nearest dollar the cost of removing, from 1 tonne of coolant,

(i) 1% of the impurities, (ii) 5% of the impurities,

(iii) 50% of the impurities, (iv) 95% of the impurities.

(b) Find an expression for the rate of change of C with respect to p.

15. A manufacturing firm produces and sells x items of a certain product. The total cost of producing these x items is \$C, with C given by

$$C = 5x + 120\sqrt{x} + 5000.$$

(a) Find the total cost of producing (i) 16 items,
(ii) 400 items.

(b) Find the average production cost per item when (i) $x = 16$,
(ii) $x = 400$.

(c) Find an expression for the marginal cost, $\frac{dC}{dx}$.

(d) Use your answer to (c) to determine the approximate cost of producing one more item at the stage in the production when (i) $x = 16$,
(ii) $x = 400$.

(e) Compare your answers for (d) part (i) to C(17) – C(16)
and for (d) part (ii) to C(401) – C(400).

16. Clearly showing your use of calculus, and in particular the second derivative test, determine the coordinates and nature of any stationary points on the graph of

$$y = (2x + 5)^3 + \frac{54}{x}.$$